



Serre Duality on Complex Manifolds

Pranav Verma

A thesis in partial fulfillment of the degree of

Bachelor of Science (Research) in Mathematics

Supervised by: Prof. Pradip Kumar

Department of Mathematics, Shiv Nadar Institution of
Eminence

April 2025

Abstract

The duality theorem of Serre aims at establishing an isomorphism between the cohomological groups over the sheaf of holomorphic forms on a complex vector bundle and the associated group over the dual vector bundle. Several results of ubiquitous importance are required along the way to establish the duality theorem. This thesis explores all the tools that lead to an elegant proof of the Serre duality.

Acknowledgements

I convey my sincere gratitude to my advisor, Prof. Pradip Kumar for all his support throughout the course of this thesis. Our discussions of the subject matter have been very fruitful, and this has fundamentally contributed to my understanding of the broad language of Algebraic Geometry. I would also like to extend my gratitude to Prof. Amber Habib for our discussions on the core ideas of Algebraic Topology in the early phases of this thesis.

Contents

1	Notations and assumptions	1
2	The Hodge-* operator	2
2.1	Basis of the exterior algebra	2
2.2	Hodge-* on general vector spaces	3
2.2.1	On complexified vector spaces and complex structures	4
2.2.2	The conjugate Hodge-* operator	5
2.3	The $\bar{*}_E$ operator on general bundles	5
3	Harmonic theory on complex manifolds	6
3.1	The Hodge inner product	6
3.2	The adjoint d^* operator	7
3.2.1	Constructing the conjugate \bar{d}^* operator	7
3.3	Space of harmonic forms $\mathcal{H}^{p,q}(X, E)$	8
3.3.1	Orthogonal projection of $\varepsilon^{p,q}(X, E)$ onto harmonic subspaces	8
3.3.2	$\mathcal{H}^{p,q}(X, E) \cong H^{p,q}(X, E)$	8
4	The Dolbeault Cohomology	10
4.1	Long exact sequences from short sequences	10
4.2	Dolbeault's theorem	13
4.2.1	Dolbeault cohomology	13
4.2.2	Associated long exact cohomology sequence on holomorphic forms	14
5	The Serre Duality	16
5.1	Statement	16

6 Cohomological dimensions of pullback bundles: possible directions to a new result	17
7 References	19
References	19

Chapter 1

Notations and assumptions

The basic setting throughout this thesis is a complex manifold X and a holomorphic vector bundle $\pi : E \rightarrow X$ over it. We assume familiarity with some sheaf-theoretic definitions of r -differential forms $\varepsilon^r(X, \wedge^r T^*(X))$, the complexification of the cotangent bundle, $\varepsilon^r(X, \wedge^r T^*(X) \otimes \mathbb{C})$ and the associated (p, q) -forms $\varepsilon^{p,q}(X, \wedge^{p,q} T^*(X) \otimes \mathbb{C})$. We likewise assume the definition of the generalized r -differential forms with coefficients in the bundle E , given by $\varepsilon^r(X, \wedge^r T^*(X) \otimes E)$ and the (p, q) -forms, $\varepsilon^{p,q}(X, \wedge^{p,q} T^*(X) \otimes E)$. All constructions and ideas leading to the Serre duality revolve around these definitions.

For clarity, we use the following notation:

- $\varepsilon^r(X) := \varepsilon^r(X, \wedge^r T^*(X))$
- $\varepsilon^{p,q}(X) := \varepsilon^{p,q}(X, \wedge^{p,q} T^*(X) \otimes \mathbb{C})$
- $\varepsilon^r(X, E) := \varepsilon^r(X, \wedge^r T^*(X)_c \otimes E)$
- $\varepsilon^{p,q}(X, E) := \varepsilon^{p,q}(X, \wedge^{p,q} T^*(X)_c \otimes E)$.

Chapter 2

The Hodge-* operator

The Hodge-* is a linear operator that uniquely maps a k -form on any vector space V of dimension n to an $(n - k)$ -form on the same space. The Hodge-* operator can be extended to act on differential forms over real manifolds or the complex structure on a complex manifold X and more generally, forms with coefficients in a holomorphic bundle E over X . This chapter shall explore all these ideas, and certain important properties of the Hodge-* operator.

2.1 Basis of the exterior algebra

Let (V, \langle, \rangle) be a real vector space of dimension n and let $\mathcal{E} = \{e_1, \dots, e_n\}$ be an orthonormal basis of V . Consider the exterior algebra of degree k , denoted by $\wedge^k V$ for some $k \leq n$. We then claim the following.

Theorem 2.1. *There exists an inner product $\langle, \rangle_{\wedge^k V}$ on $\wedge^k V$ induced from \langle, \rangle such that $\mathcal{E}^{(k)} := \{e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$ forms an orthonormal basis for $\wedge^k V$.*

Proof. We first construct an inner product on $\otimes^k V$ induced by \langle, \rangle . Specifically, for two degree k tensors comprising the basis vectors, define

$$\langle e_{i_1} \otimes \dots \otimes e_{i_k}, e_{j_1} \otimes \dots \otimes e_{j_k} \rangle_{\otimes^k V} := \prod_{l=1}^k \langle e_{i_l}, e_{j_l} \rangle.$$

This definition shows that $\mathcal{E}^{(k)}$ is orthonormal due to the orthonormality of \mathcal{E} . This can now

be extended to define an inner product $\langle, \rangle_{\wedge^k V}$ on $\wedge^k V$:

$$\langle e_{i_1} \wedge \dots \wedge e_{i_k}, e_{j_1} \wedge \dots \wedge e_{j_k} \rangle_{\wedge^k V} = \sum_{\sigma \in S_k} \sum_{\tau \in S_k} \text{sgn}(\sigma) \text{sgn}(\tau) \langle e_{i_{\sigma(1)}} \otimes \dots \otimes e_{i_{\sigma(k)}}, e_{j_{\tau(1)}} \otimes \dots \otimes e_{j_{\tau(k)}} \rangle_{\otimes^k V}$$

For orthonormality we hence define $\langle, \rangle_{\wedge^k V} := (1/k!) \langle, \rangle_{\otimes^k V}$. This is the required induced inner product on $\wedge^k V$ from the underlying space V .

2.2 Hodge-* on general vector spaces

We shall define the Hodge-* operator first for the exterior algebra of some degree k on a real inner product space (V, \langle, \rangle) . We let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V . Denote $e_1 \wedge \dots \wedge e_n = \mathcal{V}_f$ to be an oriented volume form on $\wedge^k V$. Then, the Hodge-* operator is a linear operator:

$$* : \wedge^k V \rightarrow \wedge^{n-k} V \text{ defined by } * e_{i_1} \wedge \dots \wedge e_{i_k} = \pm e_{j_1} \wedge \dots \wedge e_{j_{n-k}}.$$

Here, $\{j_1, \dots, j_{n-k}\} = \{i_1, \dots, i_k\}^c$. The sign is determined upto permutation based on the volume form \mathcal{V}_f .

The Hodge-* operator can now be extended to general differential k - forms as follows. For two differential forms $\alpha = \sum'_{|I|=k} \alpha_I e_I$ and $\beta = \sum'_{|J|=k} \beta_J e_J$, a simple computation gives:

$$\begin{aligned} \alpha \wedge * \beta &= \sum'_{|I|, |J|} \alpha_I \beta_J e_I \wedge * e_J = \sum'_{|I|} \alpha_I \beta_I \mathcal{V}_f \\ &= \langle \alpha, \beta \rangle_{\wedge^k V} \mathcal{V}_f. \end{aligned}$$

where $\langle \alpha, \beta \rangle_{\wedge^k V} = \sum'_{|I|=k} \alpha_I \beta_I$ is the induced inner product from \langle, \rangle , as discussed in section 1.1. The definition of the Hodge-* operator gives its most important property; namely, it is independent of the orthonormal basis chosen. This is true since the transformation under two sets of orthonormal bases has determinant 1, leaving the Hodge-* operator unchanged.

¹Here, $\sum'_{|I|=k} \alpha_I e_I := \sum_{1 \leq i_1 < \dots < i_k \leq n} \alpha_{i_1 \dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k}$ where $I = \{i_1, \dots, i_k\}$.

2.2.1 On complexified vector spaces and complex structures

The definition of the Hodge-* operator on a real vector space V can be extended to its complexification $V \otimes \mathbb{C}$. Specifically, define $\alpha \wedge * \bar{\beta} := \sum'_{|I|} \alpha_I \bar{\beta}_I \mathcal{V}_f$, giving $\langle \alpha, \beta \rangle = \sum'_{|I|} \alpha_I \bar{\beta}_I$. One step further, specifically consider the complex structure \mathcal{J} with $\mathcal{J}^2 = -I$ on the complexified cotangent bundle $T^*(X) \otimes \mathbb{C}$ of a complex manifold X which splits into the $\pm i$ eigenspaces:

$$T^*(X) \otimes \mathbb{C} = T^{1;0*}(X) \oplus T^{0;1*}(X)$$

where $T^{1;0*}(X)$ is the $+i$ eigenspace and $T^{0;1*}(X)$ is the $-i$ eigenspace. It can be seen that $dz_k := dx_k + idy_k$ and $d\bar{z}_k := dx_k - idy_k$ forms the respective basis of these eigenspaces, given that the complex structure smoothly assigns $\mathcal{J}(dx_k) = -dy_k$ and $\mathcal{J}(dy_k) = dx_k$. In this case, for a (p, q) -form $\alpha \in \wedge^{p;q} T^*(X) \otimes \mathbb{C}$,

$$\alpha = \sum'_{|I|=p, |J|=q} \alpha_{IJ} dz_I \wedge d\bar{z}_J \implies * \alpha := \sum'_{|I^c|=n-p, |J^c|=n-q} \alpha_{IJ} dz_{I^c} \wedge d\bar{z}_{J^c}.$$

Extending this to the sheaf of sections of (p, q) -forms, we have: $* : \varepsilon^{p;q}(X) \rightarrow \varepsilon^{n-p;n-q}(X)$ defined by

$$f(x) = \sum'_{|I|=p, |J|=q} f_{IJ}(x) dz_I \wedge d\bar{z}_J \implies * f(x) := \sum'_{|I^c|=n-p, |J^c|=n-q} f_{IJ}(x) dz_{I^c} \wedge d\bar{z}_{J^c}.$$

This leads us to the first main result in this thesis:

Theorem 2.2. $* : \varepsilon^{p;q}(X) \rightarrow \varepsilon^{n-p;n-q}(X)$ is an isomorphism.

Proof. To prove this, we first define the $**$ operator on a general k -form $v_1 \wedge \dots \wedge v_k \in \wedge^k V$, where V is a vector space of dimension n . Define $** : \wedge^k V \rightarrow \wedge^k V$ as:

$$** (v_1 \wedge \dots \wedge v_k) = *(*(v_1 \wedge \dots \wedge v_k)).$$

A simple computation shows that $** = (-1)^{k(n-k)}$. So, extending by linearity, $**$ can be used to show that $*$ is an isomorphism. Specifically, suppose $*f = *g$. This would mean that $*(f) = *(g)$, i.e., $**f = **g$. Since we have (p, q) forms, we get $(-1)^{p(n-p)+q(n-q)} f = (-1)^{p(n-p)+q(n-q)} g$, yielding $f = g$. Hence, $*$ is injective. To show that $*$ is surjective, pick an element $v = \sum'_{|I|=n-p, |J|=n-q} \alpha_{IJ}(x) dz_I \wedge d\bar{z}_J$, where α_{IJ} is a local section of the cotangent bundle. Hence, define $\alpha(x) = \sum'_{|I^c|=p, |J^c|=q} \alpha_{IJ}(x) dz_{I^c} \wedge d\bar{z}_{J^c}$ so that $*\alpha = v$.

2.2.2 The conjugate Hodge-* operator

On a complex manifold X , consider $f \in \varepsilon^r(X)$. Then, define the conjugate Hodge-* operator as:

$$\bar{*}(f) := * \bar{f}.$$

Here, the conjugation acts on the coefficients of the differential form.

2.3 The $\bar{*}_E$ operator on general bundles

We shall now describe the generalized Hodge-* operator on differential forms with coefficients from a bundle over a complex manifold. Let $\pi : E \rightarrow X$ be a Hermitian holomorphic vector bundle, and denote E' to be the dual vector bundle. Then, for $\alpha \in \wedge^r T^*(X)_c \otimes E$, define:

$$\bar{*}_E : \wedge^r T^*(X)_c \otimes E \rightarrow \wedge^{2n-r} T^*(X)_c \otimes E' \text{ by } \bar{*}_E(\alpha \otimes e) := \bar{*}(\alpha) \otimes \tau(e),$$

where $\tau : E \rightarrow E'$ is an isomorphism of bundles.

This can be extended to the sheaf of (p, q) -forms and we get the Hodge-* operator similarly defined:

$$\bar{*}_E : \varepsilon^{p,q}(X, \wedge^{p,q} T^*(X)_c \otimes E) \rightarrow \varepsilon^{n-p;n-q}(X, \wedge^{n-p;n-q} T^*(X)_c \otimes E').$$

As in Theorem 2.2, we have:

Theorem 2.3. $\bar{*}_E$ is an isomorphism.

Chapter 3

Harmonic theory on complex manifolds

The objective of Harmonic theory on complex manifolds is to establish the existence of the adjoint of the differential d -operator on the differential forms of a Hermitian holomorphic bundle E on X . This is achieved by first defining a unique inner product on the sections closely related to the Hodge- $*$ operator.

3.1 The Hodge inner product

We first construct an inner product on $\varepsilon^\bullet(X)$ as follows:

$$(\phi, \psi) := \begin{cases} \int_X \phi \wedge \bar{*}\psi & \text{if } \phi, \psi \in \varepsilon^p(X), \\ 0, & \text{if } \phi \in \varepsilon^p(X), \psi \in \varepsilon^q(X), \text{ with } p \neq q. \end{cases}$$

Theorem 3.1. *(\cdot, \cdot) defined above is indeed an inner product.*

Proof. Firstly, we have $\int_X \phi \wedge \bar{*}\psi = \int_X \langle \phi, \psi \rangle \mathcal{V}_f \geq 0$, since $\langle \cdot, \cdot \rangle$ is an inner product. Conjugate linearity follows easily. Now suppose $(\phi, \phi) = \int_X \langle \phi, \phi \rangle \mathcal{V}_f = 0$. Suppose $\phi > 0$ locally. Then, $\langle \phi, \phi \rangle = \sum'_{|I|=p} |\phi_I(x)|^2 dx_I > 0$. This would then mean that $\int_X \langle \phi, \phi \rangle \mathcal{V}_f > 0$, a contradiction.

This can be extended to $\varepsilon^\bullet(X, E)$ as follows:

$$(\phi, \psi)_E := \begin{cases} \int_X \phi \wedge \bar{*}_E \psi & \text{if } \phi, \psi \in \varepsilon^p(X, E), \\ 0, & \text{if } \phi \in \varepsilon^p(X, E), \psi \in \varepsilon^q(X, E), \text{ with } p \neq q. \end{cases}$$

3.2 The adjoint d^* operator

We know that $d : \varepsilon^\bullet(X) \rightarrow \varepsilon^{\bullet+1}(X)$ is a morphism of sheaves that locally takes any r -form to an $(r+1)$ -form. For a dimension n manifold, we obtain a short exact sequence:

$$0 \rightarrow X \hookrightarrow \varepsilon^0(X) \xrightarrow{d} \varepsilon^1(X) \rightarrow \dots \xrightarrow{d} \varepsilon^{n-1}(X) \rightarrow 0.$$

Using the Hodge inner product, we can now construct its adjoint d^* as follows. For $r < n$, consider $\phi \in \varepsilon^{r-1}(X)$ and $\psi \in \varepsilon^r(X)$. Then, $d\phi, \psi \in \varepsilon^r(X)$ gives us:

$$(d\phi, \psi) = \int_X d\phi \wedge \bar{*}\psi = \int_X d(\phi \wedge \bar{*}\psi) - (-1)^{r-1} \int_X \phi \wedge d\bar{*}\psi.$$

Then, Stokes' theorem gives us that for any closed form ω , $\int_X d\omega = 0$. Hence:

$$(d\phi, \psi) = (-1)^r \int_X \phi \wedge d\bar{*}\psi = (-1)^{r(1-n+r)} \int_X \phi \wedge \bar{*}(\bar{*}d\bar{*}\psi) = (-1)^{r(1-n+r)} (\phi, d^*\psi).$$

Hence, $d^* = (-1)^{r(1-n+r)} \bar{*}d\bar{*}$. With this, we can define the real Laplacian operator as $\Delta = dd^* + d^*d$.

3.2.1 Constructing the conjugate $\bar{\partial}^*$ operator

Since $\bar{\partial} : \varepsilon^{\bullet;\bullet}(X, E) \rightarrow \varepsilon^{\bullet;(\bullet+1)}(X, E)$, the adjoint $\bar{\partial}^*$ can be similarly constructed using the general Hodge inner product. Specifically, for $\phi \in \varepsilon^{p;q-1}(X, E)$ and $\psi \in \varepsilon^{p;q}(X, E)$, we have:

$$(\bar{\partial}\phi, \psi)_E = \int_X \bar{\partial}\phi \wedge \bar{*}_E \psi = \int_X \bar{\partial}(\phi \wedge \bar{*}_E \psi) + (-1)^{p+q} \int_X \phi \wedge \bar{\partial}\bar{*}_E \psi.$$

Now note that $\phi \wedge \bar{*}_E \psi$ is an $(n, n-1)$ -form, so $\bar{\partial}(\phi \wedge \bar{*}_E \psi)$ is an (n, n) -form. On the real underlying manifold, the action of d on $\phi \wedge \bar{*}_E \psi$ (now, a $(2n-1)$ -form) again yields a $2n$ form.

Hence, Stokes' theorem can be applied to give:

$$(\bar{\partial}\phi, \psi)_E = (-1)^{p+q} \int_X \phi \wedge \bar{\partial} \bar{*}_E \psi = - \int_X \phi \wedge \bar{*}_E (\bar{*}_E \bar{\partial} \bar{*}_E \psi).$$

Hence, $\bar{\partial}^* := -\bar{*}_E \bar{\partial} \bar{*}_E$. We can then define the complex Laplacian operator $\bar{\square} := \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$.

3.3 Space of harmonic forms $\mathcal{H}^{p,q}(X, E)$

For any p, q , the $\bar{\square}$ operator now decomposes $\varepsilon^{p,q}(X, E)$ into specific subspaces:

$$\varepsilon^{p,q}(X, E) = \mathcal{H}^{p,q}(X, E) \oplus \mathcal{H}^{p,q}(X, E)^\perp$$

where we define:

$$\mathcal{H}^{p,q}(X, E) = \{f \in \varepsilon^{p,q}(X, E) : \bar{\square}f = 0\}, \text{ the space of Harmonic } (p, q)\text{-forms,}$$

$$\mathcal{H}^{p,q}(X, E)^\perp = \{f \in \varepsilon^{p,q}(X, E) : (f, \xi)_E = 0, \xi \in \mathcal{H}^{p,q}(X, E)\}.$$

3.3.1 Orthogonal projection of $\varepsilon^{p,q}(X, E)$ onto harmonic subspaces

A result from elliptic operators show that $\dim(\mathcal{H}^{p,q}(X, E)) < \infty$. With this assumption, we can consider the orthogonal projection of $\varepsilon^{p,q}(X, E)$ onto $\mathcal{H}^{p,q}(X, E)$. Denote this projection by $H_{\bar{\square}}$.

3.3.2 $\mathcal{H}^{p,q}(X, E) \cong H^{p,q}(X, E)$

The main result of this chapter is to connect harmonic forms with certain measures of non-exactness of the short $\bar{\partial}$ sequence:

$$0 \rightarrow \Omega^\bullet(X, E) \hookrightarrow \varepsilon^{\bullet,0}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{\bullet,1}(X, E) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \varepsilon^{\bullet,n}(X, E) \rightarrow 0.$$

We define the $H^{p,q}$ space in the following manner:

$$H^{p,q}(X, E) := \frac{\text{Ker}(\varepsilon^{p,q}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q+1}(X, E))}{\text{Im}(\varepsilon^{p,q-1}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q}(X, E))}.$$

This gives us a measure of how non-exact this sequence is (This is termed as the Dolbeault cohomology group, which we shall see in Chapter 3). Now consider the following map:

$$\Phi : \text{Ker}(\varepsilon^{p,q}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q+1}(X, E)) \rightarrow \mathcal{H}^{p,q}(X, E) \text{ defined by } \Phi(f) = f_h$$

where f_h is the harmonic part of f . By the first isomorphism theorem, we have:

$$\frac{\text{Ker}(\varepsilon^{p,q}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q+1}(X, E))}{\text{Ker}(\Phi)} \cong \text{Im}(\Phi).$$

We then claim that Φ is surjective, and $\text{Ker}(\Phi) = \text{Im}(\varepsilon^{p,q-1}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q}(X, E))$. The first claim is true since, for a harmonic function $f \in \mathcal{H}^{p,q}(X, E)$, we have:

$$0 = (\bar{\square}f, f) = (\bar{\partial}^* \bar{\partial}f, f) + (\bar{\partial} \bar{\partial}^* f, f) = \|\bar{\partial}f\|^2 + \|\bar{\partial}^* f\|^2$$

yielding $\bar{\partial}f = 0$. Hence, $f \in \text{Ker}(\varepsilon^{p,q}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q+1}(X, E))$ and since f is harmonic, $\Phi(f) = f$. So, Φ is surjective. Secondly, let $\xi \in \text{Ker}(\Phi)$. Then, $\bar{\partial}\xi = 0$ and $\Phi(\xi) = 0$. This means that ξ has no harmonic part, and the decomposition then gives:

$$\xi = \bar{\square}\xi = \bar{\partial}\bar{\partial}^*\xi.$$

Now, $\bar{\partial}^*\xi = -\bar{*}\bar{\partial}\bar{*}\xi$ which is a $(p, q-1)$ -form. Hence, $\text{Ker}(\Phi) \subset \text{Im}(\varepsilon^{p,q-1}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q}(X, E))$. Conversely, let $f \in \text{Im}(\varepsilon^{p,q-1}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q}(X, E))$. Then, $f = \bar{\partial}g$, which has zero harmonic part.

Hence, we have:

Theorem 3.2. $H^{p,q}(X, E) \cong \mathcal{H}^{p,q}(X, E)$.

Chapter 4

The Dolbeault Cohomology

This chapter largely deals with a purely cohomological equivalence between the sheaf of holomorphic forms and the associated non-exactness of the $\bar{\partial}$ sequence. This equivalence is established by assuming certain Čech cohomology definitions summarized in the Appendix. The first objective is to construct long exact sequences from a set of short exact sequences in the setting of coefficients from any general sheaf, after which this shall be applied to the sheaf of holomorphic forms, giving rise to the Dolbeault theorem.

4.1 Long exact sequences from short sequences

Consider the following short exact sequence of sheaves:

$$0 \rightarrow \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \rightarrow 0.$$

The exactness condition gives $\text{Ker}(\beta) = \text{Im}(\alpha)$. This now induces a map on the p^{th} cochain group of the open cover $U = \bigcup_{i \in I} U_i$ of X , where $p \leq |I|$:

$$C^p(U, \mathcal{F}_1) \xrightarrow{\alpha} C^p(U, \mathcal{F}_2) \text{ and } C^p(U, \mathcal{F}_2) \xrightarrow{\beta} C^p(U, \mathcal{F}_3).$$

Our first claim is then the following:

Theorem 4.1. *α and β respectively commute with the coboundary operator δ between the respective cochain groups.*

Proof. We show the commutativity of α and δ . β follows similarly. Let $f \in C^p(U, \mathcal{F}_1)$. Then,

$f = (f_{i_0 i_1 \dots i_p})$ where $f_{i_0 i_1 \dots i_p} \in \mathcal{F}_1(U_{i_0} \cap \dots \cap U_{i_p})$. The Čech coboundary operator δ acts on f as:

$$\delta f = \sum_{i=0}^{p+1} (-1)^p r|_{\sigma}^{\sigma_i} f_{\sigma_i}$$

where $\sigma = U_0 \cap \dots \cap U_p$, $\sigma_i = U_0 \cap \dots \cap U_{i-1} \cap U_{i+1} \dots \cap U_p$ for each i and r is the restriction homomorphism. Then, the linearity of α gives us:

$$\alpha(\delta(f)) = \sum_{i=0}^{p+1} (-1)^p \alpha(r|_{\sigma}^{\sigma_i} f_{\sigma_i}) = \sum_{i=0}^{p+1} (-1)^p r|_{\sigma}^{\sigma_i} \alpha(f_{\sigma_i}) = \delta(\alpha(f))$$

where the sum on the right now takes place between the p^{th} order cochain groups.

This commutativity allows us to connect the respective p^{th} -cohomology groups of the different sheaves. Specifically, define the α^* operator (and likewise the β^* operator)

$\alpha^* : H^p(X, \mathcal{F}_1) \rightarrow H^p(X, \mathcal{F}_2)$ by

$$\alpha^*([f]) = [\alpha \circ f] \text{ for } f \in \text{Ker}(C^p(U, \mathcal{F}_1) \xrightarrow{\delta} C^{p+1}(U, \mathcal{F}_1)).$$

We show that α^* is well-defined. Suppose $[f_1] = [f_2]$, where $f_1, f_2 \in \text{Ker}(C^p(U, \mathcal{F}_1) \xrightarrow{\delta} C^{p+1}(U, \mathcal{F}_1))$. Then, $\delta(f_1) = \delta(f_2) = 0$ and $f_1 - f_2 = \delta \tilde{f}$ for some $\tilde{f} \in C^{p-1}(U, \mathcal{F}_1)$. Consider the following commutative diagram:

$$\begin{array}{ccc} C^{p+1}(U, \mathcal{F}_1) & \xrightarrow{\alpha} & C^{p+1}(U, \mathcal{F}_2) \\ \delta \uparrow & & \delta \uparrow \\ C^p(U, \mathcal{F}_1) & \xrightarrow{\alpha} & C^p(U, \mathcal{F}_2) \\ \delta \uparrow & & \delta \uparrow \\ C^{p-1}(U, \mathcal{F}_1) & \xrightarrow{\alpha} & C^{p-1}(U, \mathcal{F}_2) \end{array}$$

Firstly, $\alpha \circ f \in \text{Ker}(C^p(U, \mathcal{F}_2) \xrightarrow{\delta} C^{p+1}(U, \mathcal{F}_2))$, since $\delta \circ \alpha \circ f = \alpha \circ \delta \circ f = 0$, by the commutativity of α, δ . Secondly, $\alpha \circ f_1 - \alpha \circ f_2 = \alpha \circ (\delta \circ \tilde{f}) = \delta \circ (\alpha \circ \tilde{f})$. Hence, $\alpha \circ f_1 - \alpha \circ f_2 \in \text{Im}(C^{p-1}(U, \mathcal{F}_2) \xrightarrow{\delta} C^p(U, \mathcal{F}_2))$. This shows that α^* (and β^*) are well-defined maps.

We now try to define a general coboundary operator that can connect the cohomology groups between sheaves, of different orders. That is, we need a map:

$$\delta^* : H^p(X, \mathcal{F}_3) \rightarrow H^{p+1}(X, \mathcal{F}_1).$$

For this, consider the diagram:

$$\begin{array}{ccccc}
 C^p(U, \mathcal{F}_1) & \xrightarrow{\alpha} & C^p(U, \mathcal{F}_2) & \xrightarrow{\beta} & C^p(U, \mathcal{F}_3) \\
 \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
 C^{p+1}(U, \mathcal{F}_1) & \xrightarrow{\alpha} & C^{p+1}(U, \mathcal{F}_2) & \xrightarrow{\beta} & C^{p+1}(U, \mathcal{F}_3)
 \end{array}$$

To define δ^* , let $[c] \in H^p(X, \mathcal{F}_3)$. Since β is onto, there is a $b \in C^p(U, \mathcal{F}_2)$ such that $\beta(b) = c$. Then, $\delta(b)$ is in the Kernel of β , since $\beta(\delta(b)) = \delta(c) = 0$. The exactness of $C^{p+1}(U, \mathcal{F}_2)$ then gives a unique $a \in C^{p+1}(U, \mathcal{F}_1)$ such that $\alpha(a) = \delta(b)$. Lastly, $a \in \text{Ker} \delta$, since $\alpha(\delta(a)) = \delta(\alpha(a)) = \delta^2 b = 0$, and the injectivity of α would give $\delta(a) = 0$. So, we can uniquely define $\delta^* : H^p(X, \mathcal{F}_3) \rightarrow H^{p+1}(X, \mathcal{F}_1)$ by $\delta^*([c]) = [a]$ at the cohomology level.

The construction of these operators now allows us to connect a chain of short exact sequences to obtain a long sequence at the level of cohomology, as summarized below:

$$\begin{array}{ccccccc}
 H^p(X, \mathcal{F}_1) & \xrightarrow{\alpha^*} & H^p(X, \mathcal{F}_2) & \xrightarrow{\beta^*} & H^p(X, \mathcal{F}_3) & & \\
 & & & & & \searrow & \\
 & & & & & & H^{p+1}(X, \mathcal{F}_3) \\
 & & & & & \swarrow & \\
 & & & & & & H^{p+1}(X, \mathcal{F}_2) \\
 & & & & & \swarrow & \\
 & & & & & & H^{p+1}(X, \mathcal{F}_1)
 \end{array}$$

The last question is whether the long sequence is exact or not. It is sufficient to check exactness at two positions:

- At $H^p(X, \mathcal{F}_2)$:

Let $[b] \in \text{Im}(\alpha^*)$. Then, $\alpha^*[a] = [b]$ for some $[a] \in H^p(U, \mathcal{F}_1)$. We need to show that $\beta^*[b] = 0$. Since $\alpha^*[a] = [b]$, we have $b - \alpha(a) = \delta \tilde{f}$ for some $\tilde{f} \in C^{p-1}(U, \mathcal{F}_2)$. Consider $\beta(\tilde{f}) \in C^p(U, \mathcal{F}_3)$. Then, $\delta(\beta(\tilde{f})) = \beta(\delta(\tilde{f})) = \beta(b - \alpha(a)) = \beta(b) - \beta(\alpha(a)) = \beta(b)$, by the exactness of the cochain group sequence. Hence, $\beta^*[b] = 0$.

$$\begin{array}{ccccc}
 a & \xrightarrow{\alpha} & b - \alpha(a) & \xrightarrow{\beta} & \beta(b) \\
 & & \delta \uparrow & & \delta \uparrow \\
 & & \tilde{f} & \xrightarrow{\beta} & \beta(\tilde{f})
 \end{array}$$

Now suppose $[b] \in \text{Ker}(\beta^*)$. Then, $\beta(b) = \delta f$ for some $f \in C^{p-1}(U, \mathcal{F}_2)$. We need $a \in C^p(U, \mathcal{F}_1)$ such that $b - \alpha(a) \in \text{Im}(C^{p-1}(U, \mathcal{F}_2) \xrightarrow{\delta} C^p(U, \mathcal{F}_2))$. Since β is onto, there

exists $\sigma \in C^{p-1}(U, \mathcal{F}_2)$ such that $\beta(\sigma) = f$. Consider $b - \delta(\sigma)$. Then, $\beta(b - \delta(\sigma)) = \beta(b) - \beta(\delta(\sigma)) = \delta(f) - \delta(\beta(\sigma)) = 0$. Hence, by the exactness of the cochain group sequence, there exists $a \in C^p(U, \mathcal{F}_1)$ such that $b - \delta(\sigma) = \alpha(a)$. This shows that $[b] \in \text{Im}(\alpha^*)$.

$$\begin{array}{ccc} \sigma & \xrightarrow{\beta} & f = \beta(\sigma) \\ \downarrow \delta & & \downarrow \delta \\ a & \xrightarrow{\alpha} & b - \delta(\sigma) \xrightarrow{\beta} \beta(b) \end{array}$$

- At $H^p(X, \mathcal{F}_3)$:

Let $[c] \in \text{Ker}(\delta^*)$. Then, $\delta^*[c] = 0$ implies $[a] = 0$ where $a \in C^{p+1}(U, \mathcal{F}_1)$ is such that $\alpha(a) = \delta(b)$ and $b \in C^p(U, \mathcal{F}_2)$ is obtained via the surjection $\beta(b) = c$. Also, $\delta c = 0$ by definition. $[a] = 0$ means that $\alpha = \delta(\sigma)$ for some $\sigma \in C^p(U, \mathcal{F}_1)$. Then $b - \alpha(\sigma)$ is a cocycle since $\delta(b - \alpha(\sigma)) = \delta(b) - \alpha(\delta(\sigma)) = 0$. Further, $\beta(b - \alpha(\sigma)) = \beta(b) - \beta(\alpha(\sigma)) = \beta(b)$ by the exactness of the cochain group sequence. This shows $[c] \in \text{Im}(\beta^*)$. The converse follows trivially.

These proofs show that the so-formed long sequence is exact.

4.2 Dolbeault's theorem

4.2.1 Dolbeault cohomology

Define:

$$H^{p,q}(X, E) := \frac{\text{Ker}(\varepsilon^{p,q}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q+1}(X, E))}{\text{Im}(\varepsilon^{p,q-1}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q}(X, E))}$$

to be the non-exactness of the short sequence:

$$0 \rightarrow X \hookrightarrow \Omega^p(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,1}(X, E) \dots \rightarrow \varepsilon^{p,n}(X, E) \rightarrow 0.$$

This is defined as the *Dolbeault cohomology* group for any p, q .

4.2.2 Associated long exact cohomology sequence on holomorphic forms

Denote $Z^{p,q} := \text{Ker}(\varepsilon^{p,q}(X, E) \xrightarrow{\bar{\partial}} \varepsilon^{p,q+1}(X, E))$. Consider the following set of short exact sequences:

$$\begin{aligned} 0 &\longrightarrow \Omega^p \xrightarrow{i} \varepsilon^{p,0} \xrightarrow{\bar{\partial}} Z^{p,1} \longrightarrow 0 \\ 0 &\longrightarrow Z^{p,1} \xrightarrow{i} \varepsilon^{p,1} \xrightarrow{\bar{\partial}} Z^{p,2} \longrightarrow 0 \\ &\dots \\ 0 &\longrightarrow Z^{p,q} \xrightarrow{i} \varepsilon^{p,q} \xrightarrow{\bar{\partial}} Z^{p,q+1} \longrightarrow 0. \end{aligned}$$

Here, the surjectivity of the $\bar{\partial}$ operator comes from the $\bar{\partial}$ -Poincaré lemma, which we state without proof:

Theorem 4.2. (*Poincaré*). *Let α be a (p, q) $\bar{\partial}$ -closed form on a polydisc $\Delta \subset \mathbb{C}^n$, that is, $\bar{\partial}\alpha = 0$. Then, there is a $(p, q - 1)$ form β such that $\bar{\partial}\beta = \alpha$.*

We now construct the induced long exact cohomology sequences associated with the above short exact chains. Consider this component of the long exact sequence:

$$\dots \rightarrow H^{q-1}(X, \varepsilon^{p,0}) \rightarrow H^{q-1}(X, Z^{p,1}) \rightarrow H^q(X, \Omega^p) \rightarrow H^q(X, \varepsilon^{p,0}) \rightarrow \dots$$

We now make use of another theorem, which we state without proof:

Theorem 4.3. *$H^r(X, \varepsilon^{p,q}) = 0$ for $r > 0$ (Such sheaves are called fine sheaves).*

Using this theorem, the long exact cohomology sequence becomes:

$$\dots \rightarrow 0 \rightarrow H^{q-1}(X, Z^{p,1}) \rightarrow H^q(X, \Omega^p) \rightarrow 0 \rightarrow \dots$$

The first isomorphism theorem and the exactness of the sequence gives us:

$$H^q(X, \Omega^p) \cong H^{q-1}(X, Z^{p,1}).$$

This can be continued throughout the components of the long exact sequence, giving:

$$H^q(X, \Omega^p) \cong H^{q-1}(X, Z^{p,1}) \cong \dots \cong H^1(X, Z^{p,q-1}) \cong H^0(X, Z^{p,q}) / \bar{\partial}H^0(X, Z^{p,q-1}) = H^{p,q}.$$

Hence, we have proved the following important theorem by Dolbeault:

Theorem 4.4. (*Dolbeault*). $H^{p,q}(X, E) \cong H^q(X, \Omega^p(E))$. In other words, the cohomology on the sheaf of p -holomorphic forms is equivalent to the Dolbeault cohomology.

Chapter 5

The Serre Duality

The Serre Duality deals with the unification of all the constructions that we have seen in the previous chapters. While the theorem is for compact complex manifolds, Serre's proof also included a generalization to the non-compact version. The main objective of this chapter is to provide a proof in the compact setting.

5.1 Statement

Theorem 5.1. *Let X be a compact complex manifold of complex dimension n and let $\pi : E \rightarrow X$ be a holomorphic vector bundle. Denote E' as the dual bundle. Then, $H^q(X, \Omega^p(E)) \cong H^{n-q}(X, \Omega^{n-p}(E'))$.*

Proof. The general Hodge-* operator, $\bar{*}_E$, the Harmonic projection $H_{\bar{\square}}$, the isomorphism of harmonic subspaces with the cohomology group, and Dolbeault's theorem together give the required isomorphism:

$$\begin{array}{ccc}
 \varepsilon^{p,q}(X, E) & \xrightarrow{\bar{*}_E} & \varepsilon^{n-p,n-q}(X, E') \\
 \downarrow H_{\bar{\square}} & & \downarrow H_{\bar{\square}} \\
 \mathcal{H}^{p,q}(X, E) & \xrightarrow{\bar{*}_E} & \mathcal{H}^{n-p,n-q}(X, E') \\
 \downarrow \cong & & \downarrow \cong \\
 H^{p,q}(X, E) & & H^{n-p,n-q}(X, E') \\
 \downarrow \cong & & \downarrow \cong \\
 H^q(X, \Omega^p(E)) & & H^{n-q}(X, \Omega^{n-p}(E')).
 \end{array}$$

Chapter 6

Cohomological dimensions of pullback bundles: possible directions to a new result

STATEMENT:

Let $\pi : Y \rightarrow X$ be a cyclic branched covering of compact Riemann surfaces with branch divisor $B \subset X$ and ramification divisor $R \subset Y$. Let L be a line bundle on X such that $K_X^{\otimes 2} \otimes L^{-1}$ is effective.

We aim to determine necessary and sufficient conditions on the degree n of π and the branching data (i.e. the divisor B) such that the pullback bundle π^*L on Y satisfies

$$h^0(Y, \pi^*L) = h^0(X, L).$$

Our attempt at the proof, in fact, showed no possibility of obtaining the hypothesized equality.

The details are as follows.

The decomposition of cyclic covers shows that π^*L over Y carries roots of sections of $L^{\otimes n}$. This yields:

$$\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \left(\bigoplus_{j=1}^{n-1} L^{\otimes(-j)} \right).$$

Hence,

$$h^0(Y, \pi^*L) = h^0(X, L) + h^0(X, X \times \mathbb{C}) + \sum_{j=2}^{n-1} h^0(X, L \otimes L^{\otimes(-j)}).$$

Now, $h^0(X, X \times \mathbb{C}) \neq 0$, since non-trivial global sections can be defined on a compact Riemann surface. In this case, the equality $h^0(Y, \pi^*L) = h^0(X, L)$ that we require cannot be achieved, but sufficient criteria on the degree of π can be found which makes the summand 0. So,

consider $h^0(X, L \otimes L^{\otimes(-j)})$ for $2 \leq j \leq n-1$. Proceeding as above, a sufficient criteria to make $h^0(X, L \otimes L^{\otimes(-j)}) = 0$ is that $\deg(L \otimes L^{\otimes(-j)}) < 0$. Since $\deg(L \otimes L^{\otimes(-j)}) = \deg(L^{\otimes(1-j)})$, we have:

$$\deg(L^{\otimes(1-j)}) = (1-j)\deg(L) < 0.$$

Now, $j \geq 2 \implies$

$$\deg(L) > 0, \tag{6.1}$$

for the above criteria to be satisfied. The effectiveness of $K_X^{\otimes 2} \otimes L^{-1}$ further implies:

$$\deg(L) \leq 4g_X - 4.$$

Combining this with (1), we get $0 < \deg(L) \leq 4g_X - 4$ which gives us our first sufficient condition on the genus:

$$0 < \deg(L) \leq 4g_X - 4 \implies g_X > 1.$$

Secondly, the Riemann-Hurwitz theorem gives us:

$$2g_Y - 2 = (2g_X - 2)n + \deg(R) > n + \deg(R).$$

Hence, the two sufficient criteria for $h^0(X, L \otimes L^{\otimes(-j)}) = 0$ are:

$$g_X > 1 \text{ and } n < 2g_Y - 2 - \deg(R)$$

Hence, we can at the most obtain:

$$h^0(Y, \pi^*L) = h^0(X, L) + h^0(X, X \times \mathbb{C}) \geq h^0(X, L).$$

Chapter 7

References

- [1] Wells, Raymond O. *Differential Analysis on Complex Manifolds*, Springer, 1979.
- [2] Tu, Loring. *Introduction to Manifolds*, Springer, 2008.
- [3] Narasimhan, M.S. *Compact Riemann Surfaces*.
- [4] Kobayashi, Shoshichi. *Differential Analysis of Complex Vector Bundles*, Princeton Legacy Library, 1987.